Abstract Algebra I
Problem Set 1: Groups

Example style questions are marked with (*).

Groups

Question 1. Find the inverse transformations of all symmetries of an equilateral triangle.

Question 2. (*) Prove in any group G that:

- the identity element is unique;
- the inverse of the identity is the identity $e^{-1} = e$;
- the inverse g^{-1} of an element g is unique.

Question 3. Prove in any group that

$$(a_1 a_2 \dots a_n)^{-1} = a_n^{-1} \dots a_2^{-1} a_1^{-1}.$$

(Socks are put on *before* shoes, but are taken off *after* shoes!)

Question 4. (*) Suppose $a^2 = e$ for all elements $a \in G$. Prove that G is abelian.

Question 5. Find the orders of all elements in the group of symmetries of a regular triangle, square, and tetrahedron. Find the orders of all elements in the cyclic group of order 7.

Question 6. Suppose the order of an element $g \in G$ is p, a prime. Let $n \in \mathbb{Z}$. Prove that either $a^m = e$ or the order of a^m is p.

Question 7. Let $g \in G$ be an element with infinite order. Prove $a^i = a^j$ if and only if $i = j$.

Subgroups

Question 8. Verify that $\{e\}$ and G are both subgroups of the group G. These are called the *trivial subgroups*.

Question 9. (*) Let $\mathbb{Z}_n$ denote the group of integers $\{0, \dots, n-1\}$ defined with addition modulo n. Find all generators of $\mathbb{Z}_n$. Prove that:

- all subgroups of $\mathbb{Z}_n$ have the form $\{e, a^d, a^{2d}, \dots, a^{n-d}\}$, where d divides n and a is a generator of $\mathbb{Z}_n$;
- $\mathbb{Z}$ has an infinite number of subgroups; and
- the intersection of an arbitrary number of subgroups of a group G is itself a subgroup.